

# GEOMETRIC FORMULAS FOR SMALE INVARIANTS OF CODIMENSION TWO IMMERSIONS

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**ABSTRACT.** We give three formulas expressing the Smale invariant of an immersion  $f$  of a  $(4k-1)$ -sphere into  $(4k+1)$ -space. The terms of the formulas are geometric characteristics of any generic smooth map  $g$  of any oriented  $4k$ -dimensional manifold, where  $g$  restricted to the boundary is an immersion regularly homotopic to  $f$  in  $(6k-1)$ -space.

The formulas imply that if  $f$  and  $g$  are two non-regularly homotopic immersions of a  $(4k-1)$ -sphere into  $(4k+1)$ -space then they are also non-regularly homotopic as immersions into  $(6k-1)$ -space. Moreover, any generic homotopy in  $(6k-1)$ -space connecting  $f$  to  $g$  must have at least  $a_k(2k-1)!$  cusps, where  $a_k = 2$  if  $k$  is odd and  $a_k = 1$  if  $k$  is even.

## 1. INTRODUCTION

Whitney [27] classified regular plane curves up to regular homotopy: two regular curves are regularly homotopic if and only if they have the same tangential degree. He also gave a formula for the tangential degree of a plane curve in terms of its double points.

Smale [26], generalized Whitney's result to higher dimensions: associated to each immersion  $f: S^k \rightarrow \mathbb{R}^n$ ,  $n > k$  is its Smale invariant  $\Omega(f) \in \pi_k(V_{n,k})$ , the  $k^{\text{th}}$  homotopy group of the Stiefel manifold of  $k$ -frames in  $n$ -space. Two immersions are regularly homotopic if and only if they have the same Smale invariant.

Whitney's double point formula has straightforward generalizations to sphere immersions in double dimension. The Smale invariant of an immersion  $S^k \rightarrow \mathbb{R}^{2k}$  is its algebraic number of double points (mod 2 if  $k$  is odd). Also in dimensions right below double ( $S^k \rightarrow \mathbb{R}^{2k-r}$ ,  $r = 1, 2$ ) there are double point formulas for the Smale invariant, see [7] and [5].

In small codimension, the first case after plane curves is immersions  $S^2 \rightarrow \mathbb{R}^3$ . Smale's work shows that they are all regularly homotopic. Regular homotopy classes of immersions  $S^3 \rightarrow \mathbb{R}^4$  form a group isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}$ . A description of the two characterizing integers is given by Hughes [13]. The general codimension one case  $S^n \rightarrow \mathbb{R}^{n+1}$  was studied by Kaiser [15]. In the present paper our main concern is the codimension two case. Certain codimension two immersions are especially interesting due to the following:

The groups  $\pi_{4k-1}(V_{4k+1,4k-1})$ , enumerating regular homotopy classes of immersions  $S^{4k-1} \rightarrow \mathbb{R}^{4k+1}$ , are infinite cyclic. A result of Hughes and Melvin [14] says

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that in these dimensions there exist *embeddings* which are not regularly homotopic to the standard embedding. Therefore, in contrast to the case of high codimension, the Smale invariant can not be expressed solely through the self intersection. However, there are still geometric formulas for Smale invariants:

**Theorem 1.** *Let  $f: S^{4k-1} \rightarrow \mathbb{R}^{4k+1}$  be an immersion and let  $\Omega(f) \in \mathbb{Z}$  be its Smale invariant. Let  $j: \mathbb{R}^{4k+1} \rightarrow \mathbb{R}^{6k-1} = \partial\mathbb{R}_+^{6k}$  denote the inclusion and let  $M^{4k}$  be any compact oriented manifold with  $\partial M^{4k} = S^{4k-1}$ . Let  $a_k = 2$  if  $k$  is odd and  $a_k = 1$  if  $k$  is even.*

- (a) *If  $g: M^{4k} \rightarrow \mathbb{R}^{6k-1}$  is a generic map such that the restriction  $\partial g$  of  $g$  to the boundary is regularly homotopic to  $j \circ f$ , then*

$$\Omega(f) = \frac{1}{a_k(2k-1)!} \left( -\bar{p}_k[\hat{M}^{4k}] + \sharp\Sigma^{1,1}(g) \right), \quad (1)$$

$$= \frac{1}{a_k(2k-1)!} \left( -\bar{p}_k[\hat{M}^{4k}] + e(\xi(g)) \right). \quad (2)$$

- (b) *If  $g: M^{4k} \rightarrow \mathbb{R}_+^{6k}$  is a generic map such that  $g^{-1}(\partial\mathbb{R}_+^{6k}) = \partial M^{4k}$  and such that the restriction  $\partial g$  of  $g$  to the boundary is a generic immersion regularly homotopic to  $j \circ f$ , then*

$$\Omega(f) = \frac{1}{a_k(2k-1)!} \left( -\bar{p}_k[\hat{M}^{4k}] + 3t(g) - 3l(g) + L(\partial g) \right). \quad (3)$$

We give brief explanations of the various terms in the equations above:

All terms appearing in (1), (2), and (3) are integers. The term  $\bar{p}_k[\hat{M}^{4k}]$  denotes the  $k^{\text{th}}$  normal Pontryagin class of the closed manifold  $\hat{M}^{4k}$ , obtained by adding a disk to  $M^{4k}$  along  $\partial M^{4k}$ , evaluated on its fundamental homology class, see Section 2.1.

The term  $\sharp\Sigma^{1,1}(g)$  in (1) is the algebraic number of cusps of  $g$  and the term  $e(\xi(g))$  in (2) is the Euler number of the cokernel bundle of the differential of  $g$  over its singularity set, see Section 2.4.

In Equation (3), the term  $L(\partial g)$  measures the linking of the double point set of  $\partial g$  with the rest of its image, see Section 2.2, the term  $t(g)$  is the algebraic number of triple points of  $g$ , and the term  $l(g)$  measures the linking of the singularity set of  $g$  with the rest of its image, see Section 2.3.

Part (a) of Theorem 1 is proved in Section 6.5 and part (b) in Section 5.2. Equation (2) was inspired by an exercise in Gromov's book [8], see Remark 9.

Let  $\mathbf{Imm}(S^n, \mathbb{R}^{n+k})$  denote the set (group) of regular homotopy classes of immersions  $S^n \rightarrow \mathbb{R}^{n+k}$ . Theorem 1 implies the following (the notation is the same as in Theorem 1):

**Corollary 1.** *The natural map  $\mathbf{Imm}(S^{4k-1}, \mathbb{R}^{4k+1}) \rightarrow \mathbf{Imm}(S^{4k-1}, \mathbb{R}^{6k-1})$  is injective. Moreover, if  $f, g: S^{4k-1} \rightarrow \mathbb{R}^{4k+1}$  are two immersions with  $\Omega(f) - \Omega(g) = c \in \mathbb{Z}$  then the algebraic number of cusps of any generic homotopy  $F: S^{4k-1} \times I \rightarrow \mathbb{R}^{6k-1}$  connecting  $j \circ f$  to  $j \circ g$  is  $c \cdot a_k(2k-1)!$ . In particular, any such homotopy has at least  $|c| \cdot a_k(2k-1)!$  cusp points.*

Corollary 1 is proved in Section 7.1.

Combining the first statement of Corollary 1 with the result of Hughes and Melvin on embeddings  $S^{4k-1} \rightarrow \mathbb{R}^{4k+1}$  mentioned above, one concludes that the inequality in the following theorem of Kervaire [16]

**Theorem (Kervaire).** *If  $2q > 3n + 1$  then every embedding  $S^n \rightarrow \mathbb{R}^q$  is regularly homotopic to the standard embedding.*

is best possible for  $n = 4k - 1$ . (This was known to Haefliger, see [9].)

As another consequence of Theorem 1, we find first order Vassiliev invariants of generic maps of 3-manifolds into  $\mathbb{R}^4$ , and of generic maps  $S^{4k-1} \rightarrow \mathbb{R}^{6k-2}$ ,  $k > 1$ , see Remark 4.

The techniques used to prove Theorem 1 allow us also to find restrictions on self intersections of immersions:

**Theorem 2.** *Let  $V^{4k-1}$ , be a 2-connected closed oriented manifold and let  $f: V^{4k-1} \rightarrow \mathbb{R}^{4k+1}$  be a generic immersion. Then there exists an integer  $d$  such that  $d \cdot f$  is a null-cobordant immersion. Let  $M^{4k}$  be a compact oriented manifold with  $\partial M^{4k} = d \cdot V^{4k-1}$ . Let  $g: M^{4k} \rightarrow \mathbb{R}_+^{4k+2}$  be a generic immersion such that  $\partial g = d \cdot f$ . Then*

$$-\langle \bar{p}_1^k, [M^{4k}, \partial M^{4k}] \rangle + (2k+1) \sharp D_{2k+1}(g) + d \cdot L_{2k}(f) = 0. \quad (4)$$

The notion  $d \cdot f$  means the connected sum of  $d$  copies of  $f$ . The term  $L_{2k}(\partial g)$  measures the linking of the  $2k$ -fold self intersection set of  $\partial g$  with the rest of its image, the term  $\sharp D_{2k+1}(g)$  is the algebraic number of  $(2k+1)$ -fold self intersection points of  $g$ , and  $\bar{p}_1$  is the square of the relative normal Euler class. For these notions, see Section 2.5. Theorem 2 is proved in Section 8.1.

For immersions  $S^{4k+1} \rightarrow \mathbb{R}^{4k+3}$  there is a mod 2-version of the Formula (3). If  $k$  odd then there is only one regular homotopy class and the corresponding formula always vanishes, see Proposition 1. If  $k$  is even then there are two regular homotopy classes and we ask whether or not in these cases the mod 2-version gives the Smale invariant, see Question 1.

## 2. CONSTRUCTIONS AND DEFINITIONS

In this section we define all the terms appearing in the theorems stated in the Introduction. These terms are all numerical characteristics naturally associated to generic maps.

**2.1. Notation and basic definitions.** We shall work in the differential category and, unless otherwise stated, all manifolds and maps are assumed to be smooth.

It will be convenient to have a notion for the closure of a punctured manifold:

**Definition 1.** *If  $X^n$  is an  $n$ -dimensional manifold with spherical boundary  $\partial X^n \simeq S^{n-1}$ , then let  $\hat{X}^n$  denote the closed manifold obtained by gluing an  $n$ -disk to  $X^n$  along  $\partial X^n$ .*

Recall that a map  $f: M^m \rightarrow N^n$  from one manifold into another is called *stable* if there exists a neighborhood  $U(f)$  of  $f$  in the space  $C^\infty(M^m, N^n)$  of smooth maps from  $M^m$  to  $N^n$  with the following property: For any  $g \in U(f)$  there exists diffeomorphisms  $h$  of the source and  $k$  of the target such that  $g = k \circ f \circ h$ .

This paper concerns the so called “nice dimensions” of Mather, see [19], where the set of stable maps is open and dense in the space of all maps.

The notion *generic map* will be used throughout the paper. In general, generic maps constitute an open dense subset of the space of all maps. In this paper, *generic shall mean stable*. The requirement that a map is generic imposes certain conditions on the following sets associated with the map:

**Definition 2.** If  $g: X^n \rightarrow \mathbb{R}^{n+k}$  is a map of a manifold, then

- (a) the subset  $\tilde{\Sigma}(g) \subset X^n$  is defined as

$$\tilde{\Sigma}(g) = \{p \in X^n : \text{rank}(dg_p) \leq n-1\},$$

and  $\Sigma(g) = g(\tilde{\Sigma}(g)) \subset \mathbb{R}^{n+k}$ .

- (b) the subset  $D(g) \subset \mathbb{R}^{n+k}$  is defined as

$$D(g) = \{q \in \mathbb{R}^{n+k} : |g^{-1}(q)| \geq 2\},$$

where  $|A|$  denotes the cardinality of the set  $A$ , and  $\tilde{D}(g) = g^{-1}(D(g)) \subset X^n$ .

- (c) for each integer  $i \geq 2$  the subset  $D_i(g) \subset \mathbb{R}^{n+k}$  is defined as

$$D_i(g) = \{q \in \mathbb{R}^{n+k} : |g^{-1}(q)| = i\},$$

and  $\tilde{D}_i(g) = g^{-1}(D_i(g)) \subset X^n$ .

In our formulas there appear certain Pontryagin numbers. We establish notation for these:

**Definition 3.** If  $X^{4k}$  is a closed oriented  $4k$ -dimensional manifold then let  $\bar{p}_k[X^{4k}]$  denote the Pontryagin number of  $X^{4k}$  which is associated to its  $k^{\text{th}}$  normal Pontryagin class.

**2.2. A triple point invariant of generic immersions.** Let  $V^{4k-1}$  be a closed oriented manifold such that  $H_{2k}(V^{4k-1}; \mathbb{Z}) = 0 = H_{2k-1}(V^{4k-1}; \mathbb{Z})$ . Let  $f: V^{4k-1} \rightarrow \mathbb{R}^{6k-1}$  be a generic (self-transverse) immersion. Then  $D(f) = D_2(f) \subset \mathbb{R}^{6k-1}$  is an embedded  $(2k-1)$ -dimensional submanifold and there is an induced orientation on  $D(f)$  (since the codimension is even).

The normal bundle of  $f$  has dimension  $2k$  and therefore it admits a nonzero section  $v$  over  $\tilde{D}(f)$ . Let  $E_0$  denote the total space of the bundle of nonzero vectors in the normal bundle of  $f$ . The homology assumptions on  $V^{4k-1}$  imply that  $H_{2k-1}(E_0; \mathbb{Z}) = \mathbb{Z}$ . Let  $[v] \in \mathbb{Z}$  be the homology class of  $v(\tilde{D}(f))$  in  $E_0$ .

For  $p \in D(f)$ , define  $w(p) = v(p_1) + v(p_2)$ , where  $f(p_1) = f(p_2) = p$ . Then  $w$  is a normal vector field of  $D(f)$  in  $\mathbb{R}^{6k-1}$ . Let  $D'_v(f)$  be a copy of  $D(f)$  shifted slightly along  $w$ . Then  $D'_v(f) \cap f(V^{4k-1}) = \emptyset$ .

**Definition 4.** (See [4], [6]) Let  $f: V^{4k-1} \rightarrow \mathbb{R}^{6k-1}$  be an immersion as above. Define

$$L(f) = \text{lk}(D'_v(f), f(S^{4k-1})) - [v],$$

where the linking number  $\text{lk}$  is computed in  $\mathbb{R}^{6k-1}$ .

The integer  $L$  as defined in Definition 4 is well defined. That is,  $L$  is independent of the choice of  $v$ .

### 2.3. Generic maps $M^{4k} \rightarrow \mathbb{R}^{6k}$ , linking numbers, and triple points.

**Remark 1.** Let  $M^{4k}$  be a compact manifold of dimension  $4k$ . If  $g: M^{4k} \rightarrow \mathbb{R}^{6k}$  is a generic map then it has the following properties:

- (a)  $D(g) = D_2(g) \cup D_3(g)$ .
- (b)  $\tilde{D}(g) \cap \tilde{\Sigma}(g) = \emptyset$ .
- (c) At a point in  $D(g)$  the self intersection is in general position.

If  $p$  is a triple point of a generic map  $g: M^{4k} \rightarrow \mathbb{R}^{6k}$  of an *oriented* manifold ( $p \in D_3(g)$ ), then Remark 1 (c) says that the three sheets of  $M^{4k}$  meeting at  $p$  intersect in general position. Therefore, the tangent space  $T_p \mathbb{R}^{6k}$  splits into a direct sum of the three oriented  $2k$ -dimensional normal spaces of the sheets. Thus, there is an orientation induced on each triple point of  $g$ .

**Definition 5.** Let  $g: M^{4k} \rightarrow \mathbb{R}^{6k}$  be a generic smooth map. Define  $t(g)$  as the algebraic number of triple points of the map  $g$ .

For the sake of the next definition we separate the cases into:

- (a) The manifold  $M^{4k}$  is closed.
- (b) The manifold  $M^{4k}$  has non-empty boundary.

In case (a), let  $g: M^{4k} \rightarrow \mathbb{R}^{6k}$  be any generic map.

Let  $\mathbb{R}_+^{6k}$  denote a closed half-space of  $\mathbb{R}^{6k}$ . In case (b), let  $g: M^{4k} \rightarrow \mathbb{R}_+^{6k}$  be any generic map such that its restriction  $\partial g$  to the boundary  $\partial M^{4k}$  is an immersion and such that  $g^{-1}(\partial \mathbb{R}_+^{6k}) = \partial M^{4k}$ .

Then  $D(g)$  is an immersed  $2k$ -dimensional submanifold of  $\mathbb{R}^{6k}$  with non-generic triple self intersections at the triple points of  $g$  and  $\Sigma(g)$  is an embedded  $(2k-1)$ -dimensional manifold. In case (a)  $\Sigma(g)$  is the boundary of  $D(g)$ , and in case (b)  $\Sigma(g)$  is a part of the boundary of  $D(g)$  (the other part is the double points of  $\partial g$ ).

Since  $\dim(\mathbb{R}^{6k}) - \dim(M^{4k}) = 2k$  is an even number, there is an induced orientation on  $D(g)$ , which in turn induces an orientation of  $\Sigma(g)$ .

Let  $\Sigma'(g)$  be a copy of  $\Sigma(g)$  shifted slightly along the outward normal vector field of  $\Sigma(g)$  in  $D(g)$ . Then  $\Sigma'(g) \cap g(M^{4k}) = \emptyset$ .

**Definition 6.** (See [24]) Let  $g$  be a map as above. Define  $l(g)$  as the linking number of  $g(M^{4k})$  and  $\Sigma'(g)$ , in  $\mathbb{R}^{6k}$  in case (a), and in  $(\mathbb{R}_+^{6k}, \partial \mathbb{R}_+^{6k})$  in case (b).

#### 2.4. Generic maps $M^{4k} \rightarrow \mathbb{R}^{6k-1}$ , cusps, and Euler classes.

**Remark 2.** Let  $M^{4k}$  be a compact manifold of dimension  $4k$ . If  $g: M^{4k} \rightarrow \mathbb{R}^{6k-1}$  is a *generic* map then it has the following properties:

- (a) For  $k > 1$ ,  $D(g) = D_2(g) \cup D_3(g)$ . For  $k = 1$ ,  $D(g) = D_2(g) \cup \dots \cup D_5(g)$ .
- (b) For  $k > 1$ ,  $\tilde{D}_3(g) \cap \tilde{\Sigma}(g) = \emptyset$ . For  $k = 1$ ,  $(\tilde{D}_4(g) \cup \tilde{D}_5(g)) \cap \tilde{\Sigma}(g) = \emptyset$ .
- (c)  $\tilde{\Sigma}(g)$  is a  $2k$ -dimensional submanifold of  $M^{4k}$ . At each point  $p \in \tilde{\Sigma}(g)$ ,  $\text{rank}(dg) = 4k - 1$ . Thus, the kernel  $\ker(dg)$  of  $dg$  is a 1-dimensional subbundle of the restriction  $TM^{4k}|_{\tilde{\Sigma}(g)}$  of the tangent bundle to  $\tilde{\Sigma}(g)$ .

Moreover, the fiber  $\ker(dg)_p \subset T_p M^{4k}$  does not lie in  $T_p \tilde{\Sigma}(g) \subset T_p M^{4k}$  for all but finitely many  $p \in \tilde{\Sigma}(g)$ . The finitely many exceptional points are called  $\Sigma^{1,1}$ -points or *cusps*.

- (d) No cusp is in  $\tilde{D}(g)$ . At a point in  $D(g) - \Sigma(g)$  the self intersection is in general position. At a point in  $D(g) \cap \Sigma(g)$  the smooth sheet (or sheets if  $k = 1$ ) of  $g(M^{4k})$  meets  $\Sigma(g)$  in general position. (Since no cusp is in  $\tilde{D}(g)$ , the tangent space of  $\Sigma(g)$  is well defined at all points in  $D(g) \cap \Sigma(g)$ ).

It is proved in [25], Appendix 1, that if  $q$  is a  $\Sigma^{1,1}$ -point of a generic map  $g: M^{4k} \rightarrow \mathbb{R}^{6k-1}$  of an *oriented* manifold then there are induced orientations on  $T_{g(q)} \mathbb{R}^{6k-1}$  and  $T_q M^{4k}$ . Taking the product of these orientations, a sign is associated to each  $\Sigma^{1,1}$ -point.

**Definition 7.** If  $g: M^{4k} \rightarrow \mathbb{R}^{6k-1}$  is a generic map of an oriented manifold then let  $\sharp \Sigma^{1,1}(g)$  denote its algebraic number of  $\Sigma^{1,1}$ -points.

**Definition 8.** If  $g: M^{4k} \rightarrow \mathbb{R}^{6k-1}$  is a generic map then let  $\xi(g)$  denote the  $2k$ -dimensional vector bundle over  $\tilde{\Sigma}(g)$ , the fiber of which over  $p \in \tilde{\Sigma}(g)$  is  $\xi(g)_p = T_{g(p)}\mathbb{R}^{6k-1}/dg(T_p M^{4k})$ .

The total space  $E(g)$  of  $\xi(g)$  is orientable (and even oriented), see Lemma 5. In this situation, the Euler number of  $\xi(g)$  is a well-defined integer (in the case when  $\tilde{\Sigma}(g)$  is non-orientable it is necessary to use homology with twisted coefficients, see Example 1 and Section 6.1.)

**Definition 9.** If  $g: M^{4k} \rightarrow \mathbb{R}^{6k-1}$  is a generic map of an oriented manifold then let  $e(\xi(g))$  denote the Euler number of the bundle  $\xi(g)$  over  $\tilde{\Sigma}(g)$  (see Definition 8).

**2.5. Self intersection points of high multiplicity.** Let  $M^{4k}$  be a compact oriented manifold of dimension  $4k$  and let  $g: M^{4k} \rightarrow \mathbb{R}^{4k+2}$  be a generic (self transverse) immersion. Then  $g$  has isolated  $(2k+1)$ -fold self intersection points. At such a point the sheets of  $M^{4k}$  meet in general position. The oriented normal spaces to the sheets of  $M^{4k}$  at  $p$  induce an orientation of  $T_p\mathbb{R}^{4k+2}$ .

**Definition 10.** For  $g$  as above let  $\sharp D_{2k+1}(g)$  denote the algebraic number of  $(2k+1)$ -fold self intersection points of  $g$ .

Let  $V^{4k-1}$  be a 2-connected closed oriented manifold and let  $f: V^{4k-1} \rightarrow \mathbb{R}^{4k+1}$  be a generic immersion. Then the  $2k$ -fold self intersection points of  $f$  form a closed 1-manifold  $D_{2k}(f) \subset \mathbb{R}^{4k+1}$ . Since  $V^{4k-1}$  is 2-connected it has a (unique up to homotopy) normal framing  $(n_1, n_2)$ . Let  $D'_{2k}(f)$  be the manifold which is obtained when  $D_{2k}(f)$  is shifted slightly along the vector field,  $p \mapsto n_1(p_1) + \dots + n_1(p_{2k})$ , for  $p \in D_{2k}(f)$ ,  $p = f(p_1) = \dots = f(p_{2k})$ .

**Definition 11.** For an immersion  $f$  as above, define

$$L_{2k}(f) = \text{lk}(f(S^{4k-1}), D'_{2k}(f)),$$

where the linking number  $\text{lk}$  is computed in  $\mathbb{R}^{4k+1}$ .

Let  $f: V^{4k-1} \rightarrow \mathbb{R}^{4k+1}$  be an immersion. Then there is a nonzero integer  $d$  such that  $d \cdot f = f \sharp \dots \sharp f$  bounds an immersion  $g: M^{4k} \rightarrow \mathbb{R}^{4k+2}$ . (See Section 8 for this fact and [6] for the connected sum operation on *generic* immersions.)

The normal bundle  $\nu_g$  of  $g$  is trivial over  $d \cdot V^{4k-1} = \partial M^{4k}$ , since  $V^{4k-1}$  is 2-connected. Moreover, its trivialization is homotopically unique. Hence, the Euler class  $\bar{e}$  of  $\nu_g$  can be considered as a class in  $H^2(M^{4k}, \partial M^{4k}; \mathbb{Z})$ . We introduce the following notation:

$$\bar{p}_1 = p_1(\nu_g) = \bar{e}^2.$$

Finally, we let  $\langle \bar{p}_1^k, [M^{4k}, \partial M^{4k}] \rangle$  denote the evaluation of  $\bar{p}_1^k$  on the orientation class  $[M^{4k}, \partial M^{4k}]$ .

### 3. SOME REMARKS ON THE MAIN RESULTS

In this section we make several remarks concerning extensions of our formulas and concerning their relations to other results.

### 3.1. The Smale invariant formulas.

**Remark 3.** Consider the first case,  $k = 1$ , in Theorem 1. We can rewrite Formulas (1), (2), and (3) using the following well-known identities for an oriented closed 4-manifold  $X$ ,

$$-\bar{p}_1[X] = p_1[X] = 3\sigma(X),$$

where  $p_1[X]$  is the Pontryagin class of the tangent bundle evaluated on the orientation class of  $X$  and  $\sigma(X)$  denotes the signature of  $X$ .

The signature of a non-closed 4-manifold  $M^4$  can still be defined, see [18]. In the general case the quadratic form on  $H_2(M^4; \mathbb{Z})$  may be degenerate but if  $M^4$  has spherical boundary then  $\sigma(M^4) = \sigma(\hat{M}^4)$ .

The resulting formulas are

$$\Omega(f) = \frac{1}{2} (3\sigma(M^4) + \sharp\Sigma^{1,1}(g)), \quad (5)$$

$$= \frac{1}{2} (3\sigma(M^4) + e(\xi(g))) \quad (6)$$

replacing Equations (1) and (2), respectively, and

$$\Omega(f) = \frac{1}{2} (3\sigma(M^4) + 3t(g) - 3l(g) + L(f)), \quad (7)$$

replacing Equation (3).

**Remark 4.** Formulas (5) and (6) give invariants of arbitrary generic maps of closed oriented 3-manifolds  $V^3$  into  $\mathbb{R}^4$ :

Fix some oriented manifold  $M^4$  such that  $\partial M^4 = V^3$ . Let  $f: V^3 \rightarrow \mathbb{R}^4$  be any generic map. Then there exists a generic map  $g: M^4 \rightarrow \mathbb{R}_+^6$  such that  $\partial g = f$ . Define

$$v(f) = \sigma(M^4) + \sharp\Sigma^{1,1}(g) = \sigma(M^4) + e(\xi(g)).$$

By Lemmas 6 and 7,  $v$  is well defined. Moreover,  $v$  does not change during homotopies through generic maps. More precisely,  $v$  does not change under homotopies avoiding maps with cusps and changes by  $\pm 1$  at cusp instances. Thus,  $v$  is a Vassiliev invariant of degree one.

In the same way we can define first order Vassiliev invariants of generic maps  $S^{4k-1} \rightarrow \mathbb{R}^{6k-2}$ : Let  $f$  be such a map. Pick a generic map  $F: D^{4k} \rightarrow \mathbb{R}_+^{6k-1}$  extending  $f$ . Define  $v(f) = \sharp\Sigma^{1,1}(F)$ . Then  $v$  is well defined by Lemma 6. Again  $v$  is an invariant of generic maps and changes by  $\pm 1$  at cusp instances.

**Remark 5.** The expressions for  $\Omega(f)$  given in Formulas (5), (6), and (7) make sense also when  $f: V^3 \rightarrow \mathbb{R}^5$  is a framed generic immersion of any oriented 3-manifold  $V^3$ :

The signature still makes sense, see Remark 3. The definition of  $L(f)$  can be extended to this case as follows:

Let  $n_1, n_2$  be the two linearly independent normal vectors in the normal bundle of  $f$ , which give its framing. The fiber of the normal bundle of  $f$  at a point  $p \in V^3$  can be identified with  $T_{f(p)}\mathbb{R}^5/df(T_p V^3)$ . Using this identification we let  $w$  be the vector field along  $D(f)$  defined by  $w(q) = n_1(p_1) + n_2(p_2)$ , where  $f(p_1) = f(p_2) = q$ . Let  $D'(f)$  be the manifold which results from pushing  $D(f)$  a small distance along  $w$ . Then  $D'(f) \cap f(V^3) = \emptyset$ .

Define  $L(f)$  as the linking number of  $D'(f)$  and  $f(V^3)$  in  $\mathbb{R}^5$ . (In the special case of a homology sphere this definition agrees with Definition 4.)

Let  $\pi^s(3)$  denote the stable homotopy group  $\pi_{N+3}(S^N)$ ,  $N$  large. If  $f: V^3 \rightarrow \mathbb{R}^5$  is framed generic immersion of any 3-manifold, then  $\Omega(f)$  (as given in Formulas (5), (6), and (7)) reduced modulo 24 gives the element in  $\pi^s(3) = \mathbb{Z}_{24}$  realized by the framed immersion  $f$ .

**Remark 6.** In higher dimensions, we can find analogs of Formulas (5), (6), and (7) when  $\hat{M}^{4k}$  is almost parallelizable. If this is the case then  $\bar{p}_k[\hat{M}^{4k}]$  can be replaced by the appropriate multiple of the signature:

$$-\bar{p}_k[\hat{M}^{4k}] = p_k[\hat{M}^{4k}] = \frac{(2k)!}{2^{2k}(2^{2k-1} - 1)B_k} \sigma(M^{4k})$$

where  $B_k$  is the  $k$ -th Bernoulli number, see [20].

**Remark 7.** Formulas (1), (2), and (3) reduced modulo the order of the image of the  $J$ -homomorphism give the element represented by the immersion  $f: S^{4k-1} \rightarrow \mathbb{R}^{4k+1}$  with its homotopically unique normal framing in  $\pi^s(4k-1)$ .

**Remark 8.** The right-hand sides of Equations (1), (2), and (3) can be altered as follows:

Replace the normal Pontryagin class of  $\hat{M}^{4k}$  by the corresponding class of the normal bundle of  $M^{4k}$  considered as an element in the relative cohomology group  $H^{4k}(M^{4k}, \partial M^{4k}; \mathbb{Z})$ . (This is possible since the normal bundle over the boundary is trivialized; it inherits a homotopically unique normal framing from the codimension two immersion  $f$ .) These altered formulas vanish for any generic  $g$  satisfying the conditions in Theorem 1. (For  $k = 1$ , Theorem 2 is one of these altered formulas.)

The proof of these facts are the same as the proofs of the formulas themselves.

**Remark 9.** Formula (2) was inspired by Exercise (c) on page 65 in Gromov's book [8]. The exercise reads:

“Let  $V$  be a closed oriented 4-manifold and let  $f: V \rightarrow \mathbb{R}^5$  be a generic  $C^\infty$ -map. Then the singularity  $\Sigma = \Sigma_f^1$  is a smooth closed surface in  $V$  such that  $\text{rank}_v df = 3$  for all  $v \in \Sigma$ . Let  $g: \Sigma \rightarrow Gr_3\mathbb{R}^5$  be the map which assigns the image  $D_f(T_v)$  (which is a 3-dimensional subspace in  $\mathbb{R}^5$ ) to each point  $v \in V$ . Prove, for properly normalized Euler form  $\omega$  in  $Gr_3\mathbb{R}^5$ , [which is a closed  $SO(5)$  invariant 2-form on the Grassmann manifold  $Gr_3\mathbb{R}^5 = Gr_2\mathbb{R}^5$ ], the equality  $\int_\Sigma g^*(\omega) = p_1(V^4)$  for a natural orientation in  $\Sigma(g)$  and for the first Pontryagin number  $p_1(V)$  of  $V$ .”

Using the notation introduced in Section 2.4, the Euler class  $g^*(\omega)$  is the Euler class of the bundle  $\xi(f)$ . Since  $\Sigma$  ( $\tilde{\Sigma}(f)$  in our notation) is sometimes non-orientable, the statement “for a natural orientation in  $\Sigma$ ” should be understood in terms of homology with local coefficients, see Section 6.1.

We give an example with (necessarily) non-orientable singularity surface:

**Example 1.** Let  $f: \mathbb{C}P^2 \rightarrow \mathbb{R}^5$  be any generic map. It follows from Lemma 5 below that the determinant bundle  $\det(T\tilde{\Sigma}(f))$  of  $\tilde{\Sigma}(f)$  is isomorphic to 1-dimensional vector bundle  $\ker(df)$  over  $\tilde{\Sigma}(f)$ . Thus the orientability of  $\tilde{\Sigma}(f)$  implies that  $\ker(df)$  is a trivial line bundle. Hence, by Lemma (A) on page 49 in [8], there exists a function  $\phi$  such that  $d\phi(\ker(df)) \neq 0$ . Then  $F: \mathbb{C}P^2 \rightarrow \mathbb{R}^5 \times \mathbb{R}$ ,  $F(x) = (f(x), \phi(x))$  is an immersion. However, since  $p_1[\mathbb{C}P^2] = 3$  is not a square,  $\mathbb{C}P^2$  does not immerse

in  $\mathbb{R}^6$ . (If it did immerse,  $p_1(\mathbb{C}P^2)$  would equal  $\bar{e}^2$ , where  $\bar{e}$  is the Euler class of the normal bundle of the immersion.) This contradiction shows that  $\tilde{\Sigma}(f)$  cannot be orientable.

### 3.2. Regular homotopy of embeddings in high codimension.

**Remark 10.** In [14], Hughes and Melvin pose the following question: “For a given  $n$ , what is the largest possible value of  $k$  such that  $\mathbf{Emb}(S^n, \mathbb{R}^k) \neq 0$ ?” Here,  $\mathbf{Emb}(S^n, \mathbb{R}^k)$  denotes the set (group) of regular homotopy classes which can be represented by embeddings. Combining Corollary 1 and the theorem of Kervaire stated in the Introduction gives the answer for  $n = 4k - 1$ :

There are infinitely many regular homotopy classes of immersions  $S^{4k-1} \rightarrow \mathbb{R}^q$  containing embeddings if  $4k + 1 \leq q \leq 6k - 1$ . If  $q \geq 6k$  then there is only one such class.

### 3.3. Bounding immersions.

**Remark 11.** Let  $V^{4k-1} \approx S^{4k-1}$  in Theorem 2. Replacing the term  $\langle \bar{p}_1^k, [M^{4k}, \partial M^{4k}] \rangle$  in Formula (4) by  $\bar{p}_1^k[\hat{M}^{4k}]$ , the equation would still hold for  $k > 1$ . However, for  $k = 1$ , the left-hand side would (using Theorem 1 (b)) equal  $d \cdot \Omega(f)$ .

This difference between  $k = 1$  and  $k > 1$  comes from the fact that if  $M^{4k}$ ,  $k > 1$  is a manifold with spherical boundary  $\partial M^{4k} = S^{4k-1}$  and  $g: M^{4k} \rightarrow \mathbb{R}_+^{4k+2}$  is an immersion, then  $\bar{p}_1(\hat{M}^{4k})$  corresponds to  $p_1(\nu_g)$ , where  $\nu_g$  is the normal bundle of  $g$ . In this case we can consider  $\bar{p}_1$  as an absolute class already in  $M^{4k}$  since

$$H^4(\hat{M}^{4k}; \mathbb{Z}) \approx H^4(M^{4k}, \partial M^{4k}; \mathbb{Z}) \approx H^4(M^{4k}; \mathbb{Z}).$$

This is not the case if  $k = 1$ , then  $H^4(M^4; \mathbb{Z}) = 0$ .

**Remark 12.** According to Remark 11, Theorem 2 should not be considered as a generalization of Theorem 1 (b). Instead it can be considered as a generalization of a theorem in [3], in which the number of triple points of a surface immersed in the upper half-space and bounding a given regular plane curve has been computed.

## 4. SMALE INVARIANTS OF EMBEDDINGS $S^{4k-1} \rightarrow \mathbb{R}^{4k+1}$

In this section we shall state a result which will be used in our proof of Theorem 1. The result is due to Hughes and Melvin [14].

Let  $f: S^{4k-1} \rightarrow \mathbb{R}^{4k+1}$  be a smooth embedding. It is a well known fact that any such  $f$  admits a Seifert-surface. That is, there exists a smooth orientable  $4k$ -manifold  $M^{4k}$ , with spherical boundary and an embedding  $g: M^{4k} \rightarrow \mathbb{R}^{4k+1}$  such that the restriction  $\partial g$  of  $g$  to  $\partial M^{4k}$  equals  $f$ .

Since  $M^{4k}$  immerses into  $\mathbb{R}^{4k+1}$  it follows that  $\hat{M}^{4k}$  is almost parallelizable. The immersion  $g$  gives a framing of the stable tangent bundle of  $\hat{M}^{4k}$  along  $M^{4k}$ . The obstruction to extending this trivialization over the added disk can on the one hand be expressed in terms of the  $k^{\text{th}}$  Pontryagin class of  $\hat{M}^{4k}$  (see [20]) and on the other hand it can be expressed in terms of the Smale invariant of  $f$ . Using this observation one can prove the following:

**Lemma 1.** *There are embeddings  $S^{4k-1} \rightarrow \mathbb{R}^{4k+1}$  which are not regularly homotopic to the standard embedding. Moreover, if  $f: S^{4k-1} \rightarrow \mathbb{R}^{4k+1}$  is an embedding and  $g: M^{4k} \rightarrow \mathbb{R}^{4k+1}$  is a Seifert-surface of  $f$  then*

$$\Omega(f) = \frac{1}{a_k(2k-1)!} p_k[\hat{M}^{4k}],$$

where  $\Omega(f)$  is the Smale invariant of  $f$ ,  $a_k = 2$  for  $k$  odd, and  $a_k = 1$  for  $k$  even.

*Proof.* As mentioned above, this is proved in [14]. Here we present a simple, slightly different proof. Let  $\mathbf{Imm}(S^{4k-1}, \mathbb{R}^{4k+1})$  denote the set of regular homotopy classes of immersions  $S^{4k-1} \rightarrow \mathbb{R}^{4k+1}$  and let  $\pi_{4k-1}(SO) \cong \mathbb{Z}$  denote the stable homotopy group of the orthogonal group.

There is a map  $\mathbf{Imm}(S^{4k-1}, \mathbb{R}^{4k+1}) \rightarrow \pi_{4k-1}(SO)$ : Given an immersion, lift it to  $\mathbb{R}^N$ ,  $N$  large. This immersion has a homotopically unique (normal) framing. (The unique framing of the immersion in  $\mathbb{R}^{4k+1}$  plus the trivial framing of  $\mathbb{R}^{4k+1}$  in  $\mathbb{R}^N$ .) Deform this lifted immersion to the standard embedding and compare the induced framing with the standard one (obtained from the standard embedding of  $S^{4k-1} \subset \mathbb{R}^{4k} \subset \mathbb{R}^{4k+1} \subset \mathbb{R}^N$ ).

This map has an inverse: Via the Hirsch lemma, any given framing on the standard embedding  $S^{4k-1} \rightarrow \mathbb{R}^N$  can be used to push the framed immersion down to an immersion  $S^{4k-1} \rightarrow \mathbb{R}^{4k+1}$ , the regular homotopy class of which is well-defined.

It is well known that  $\pi_{4k-1}(SO) = \text{Vect}(S^{4k})$ , where  $\text{Vect}(S^{4k})$  denotes the group of stable equivalence classes of vector bundles on  $S^{4k}$ . If  $\eta \in \text{Vect}(S^{4k})$  is a stable bundle then let  $[\eta]$  denote the corresponding element in  $\pi_{4k-1}(SO)$  and if  $f: S^{4k-1} \rightarrow \mathbb{R}^{4k+1}$  is an immersion then let  $\eta_f$  denote the stable bundle corresponding to  $f$ .

Let  $p_k: \text{Vect}(S^{4k}) \rightarrow \mathbb{Z}$  denote the map defined by  $\eta \mapsto \langle p_k(\eta), [S^{4k}] \rangle$ . Lemma 2 in [20] says that  $\eta \mapsto \langle p_k(\eta), [S^{4k}] \rangle = a_k(2k-1)![\eta]$ , where  $a_k = 2$  if  $k$  is odd and  $a_k = 1$  if  $k$  is even.

Consider the map  $\mathbf{Imm}(S^{4k-1}, \mathbb{R}^{4k+1}) \rightarrow \mathbb{Z}$  defined as the composition of the above two, i.e.  $f \mapsto p_k(\eta_f)$ . Since the first map is onto, this composition is a map of  $\mathbb{Z}$  onto  $a_k(2k-1)! \cdot \mathbb{Z}$ . Hence, it is  $\pm a_k(2k-1)! \cdot \Omega$ , where  $\Omega$  is the Smale invariant. That is,  $\pm a_k(2k-1)! \cdot \Omega(f) = \langle p_k(\eta_f), [S^{4k}] \rangle$ .

Consider the standard embedding of  $S^{4k-1}$  in  $\mathbb{R}^N$  with framing induced from an immersion  $f: S^{4k-1} \rightarrow \mathbb{R}^{4k+1}$ . Assume that it bounds a framed immersion  $F: M^{4k} \rightarrow \mathbb{R}_+^{N+1}$ . Let  $\nu$  denote the stable normal bundle of the closed manifold  $\hat{M}^{4k}$ . It is straightforward to find a map  $g: \hat{M}^{4k} \rightarrow S^{4k}$  of degree 1 such that  $g^*(\eta_f) = \nu$ . Thus,

$$\langle p_k(\nu), [\hat{M}^{4k}] \rangle = \langle p_k(\eta_f), [S^{4k}] \rangle.$$

Note that  $p_k[\hat{M}^{4k}] = -p_k(\nu)$  (since  $\hat{M}^{4k}$  is almost parallelizable). Hence,  $p_k[\hat{M}^{4k}] = \pm a_k(2k-1)! \cdot \Omega(f)$ .

Especially, one can apply this when  $f$  is an embedding and  $M^{4k}$  is a Seifert surface of  $f$ . This proves the formula in Lemma 1.

The existence statement is proved by noting that there exist almost parallelizable  $4k$ -manifolds  $\hat{X}^{4k}$ , with  $p_k[\hat{X}^{4k}] \neq 0$ , and with a handle decomposition consisting of one 0-handle,  $2k$ -handles, and one  $4k$ -handle. Let  $X^{4k}$  be a punctured  $\hat{X}^{4k}$ . Then  $X^{4k}$  immerses into  $\mathbb{R}^{4k+1}$  by the Hirsch lemma. Its  $2k$ -skeleton actually embeds in  $\mathbb{R}^{4k+1}$  by general position, but  $X^{4k}$  is a regular neighborhood of this  $2k$ -skeleton in  $\hat{X}^{4k}$  so  $X^{4k}$  also embeds. The boundary of an embedded  $X^{4k}$  is an embedded sphere with non-trivial Smale invariant.  $\square$

## 5. GENERIC $4k$ -MANIFOLDS IN $6k$ -SPACE AND

## THE PROOF OF THEOREM 1 (b)

In this section we present some lemmas on generic maps and use these to prove Theorem 1 (b).

**5.1. Generic  $4k$ -manifolds in  $6k$ -space.** We shall show that the difference of triple points and singularity linking of a generic map of a closed oriented  $4k$ -manifold in  $6k$ -space is a cobordism invariant. The proof will use the following well known fact about linking numbers.

**Lemma 2.** *Let  $X$  and  $Y$  be relative oriented cycles of complementary dimensions in  $\mathbb{R}^n \times I$ , where  $I$  is the unit interval. Let  $\partial_i X$  and  $\partial_i Y$  be  $X \cap \mathbb{R}^n \times \{i\}$  and  $Y \cap \mathbb{R}^n \times \{i\}$ , respectively, for  $i = 0, 1$ . Assume that  $\partial_i X \cap \partial_i Y = \emptyset$  for  $i = 0, 1$  and that the intersection of  $X$  and  $Y$  is transverse. Then*

$$X \bullet Y = \text{lk}(\partial_1 X, \partial_1 Y) - \text{lk}(\partial_0 X, \partial_0 Y),$$

where  $X \bullet Y$  denotes the intersection number of  $X$  and  $Y$ . □

**Lemma 3.** *Let  $M_0^{4k}$  and  $M_1^{4k}$  be two closed and oriented manifolds. Let  $k_i: M_i^{4k} \rightarrow \mathbb{R}^{6k}$ ,  $i = 0, 1$ , be generic smooth maps. Assume that there exists an oriented cobordism  $K: W^{4k+1} \rightarrow \mathbb{R}^{6k} \times I$ , where  $I$  is the unit interval, joining  $k_0$  to  $k_1$ . Then*

$$t(k_0) - l(k_0) = t(k_1) - l(k_1).$$

(For notation, see Definitions 5 and 6.)

*Proof.* We may assume that the map  $K$  is generic. Then  $K$  does not have any 4-fold self intersection points. The triple points of  $K$  form a 1-manifold  $D_3(K)$  and there is an induced orientation of  $D_3(K)$ . The boundary of  $D_3(K)$  consists of three types of points:

- (a) Triple points of  $k_1$ ,
- (b) triple points of  $k_0$ , and
- (c) those double points of  $K$  which are also singular values. (That is, points in  $\Sigma(K) \cap D(K)$ .) At such a double point a nonsingular sheet of  $W^{4k+1}$  meets the  $2k$ -dimensional singular locus  $\Sigma(K)$  of  $K$  transversely. (In other words, it is a double point of type  $\Sigma^{1,0} + \Sigma^0$ .)

There is an induced orientation on the manifold  $D_2(K)$  of nonsingular double points of  $K$ . Therefore, there is an induced orientation on its boundary  $\Sigma(K)$ . Hence, there are also induced orientations on the points of type (c) above.

Note that a point  $p$  of type (a) is a positive triple point of  $k_1$  if and only if the orientation of  $D_3(K)$  close to  $p$  points towards  $p$  (out of  $D_3(K)$ ), that a point  $q$  of type (b) is a positive triple point of  $k_0$  if and only if the orientation points away from  $q$  (into  $D_3(K)$ ), and that a point  $r$  of type (c) has positive sign if and only if the orientation points towards  $r$ .

Let  $A(K)$  denote the algebraic number of points of type (c). Since the triple curves of  $K$  give a cobordism between the points of type (c) and those of type (a) or (b), it follows that

$$A(K) = t(k_1) - t(k_0). \tag{8}$$

Now consider the  $2k$ -dimensional manifold  $\Sigma(K)$  of singular values of  $K$ . The boundary of  $\Sigma(K)$  consists of  $\Sigma(k_1)$  and  $\Sigma(k_2)$ . Let  $n$  be the outward normal vector

field of  $\Sigma(K)$  in  $D_2(K)$ . Pushing  $\Sigma(K)$  along  $n$  we obtain an oriented manifold  $\Sigma'(K)$ .

Close to each point  $p$  of type (c) above,  $\Sigma'(K)$  intersects  $K(W^{4k+1})$  at one point with local intersection number equal to the sign of  $p$ ,  $\Sigma'(K)$  does not intersect  $K(W^{4k+1})$  in other points and the boundary of  $\Sigma'(K)$  is  $\Sigma'(k_1) - \Sigma'(k_0)$ . Therefore, by Lemma 2,

$$A(K) = l(k_1) - l(k_0).$$

□

**Lemma 4.** *Let  $M^{4k}$  be a closed and oriented manifold and let  $h: M^{4k} \rightarrow \mathbb{R}^{6k}$  be a generic smooth map. Then*

$$-\bar{p}_k[M^{4k}] + 3t(h) - 3l(h) = 0.$$

(For notation, see Definitions 5 and 6.)

*Proof.* For immersions  $h$ ,  $l(h) = 0$  and Lemma 4 is a theorem of Herbert, see [11].

Let  $\mathbf{Imm}^{SO}(4k, 2k)$  denote the cobordism group of immersions of oriented  $4k$ -dimensional manifolds in  $\mathbb{R}^{6k}$ . Let  $\Omega_{4k}(\mathbb{R}^{6k})$  be the cobordism group of generic maps of oriented  $4k$ -manifolds into  $\mathbb{R}^{6k}$ . Then, of course,  $\Omega_{4k}(\mathbb{R}^{6k}) \approx \Omega_{4k}$ , where  $\Omega_{4k}$  is the cobordism group of oriented  $4k$ -manifolds.

A theorem of Burlet [2] says that the cokernel of the natural map  $\mathbf{Imm}^{SO}(4k, 2k) \rightarrow \Omega_{4k}(\mathbb{R}^{6k})$  is *finite*. (Sketch of proof of Burlet's theorem: Note that  $\mathbf{Imm}^{SO}(4k, 2k) \approx \pi_{6k}^s(MSO(2k)) \approx \pi_{6k+K}(\Sigma^K MSO(2k))$  and that  $\Omega_{4k} \approx \pi_{4k+K}(MSO(K)) \approx \pi_{6k+K}(MSO(2k+K))$  for  $K$  sufficiently large. Apply Serre's theorem, saying that the rational stable Hurewicz homomorphism  $\pi_i^s(X) \otimes \mathbb{Q} \rightarrow H_i(X; \mathbb{Q})$  is an isomorphism for any space  $X$ , pass to (co)homology, and use the Thom isomorphism and the well known ring  $H^*(BSO(m); \mathbb{Q})$ .)

Now, by Lemma 3, the difference  $3t(g) - 3l(g)$  is invariant under cobordism and hence gives rise to a homomorphism  $\Omega_{4k}(\mathbb{R}^{6k}) \rightarrow \mathbb{Z}$ . Denote it  $\Lambda$ . By Herbert's theorem  $\Lambda$  equals  $\bar{p}_k$  on the image of the group  $\mathbf{Imm}^{SO}(4k, 2k)$  in  $\Omega_{4k}(\mathbb{R}^{6k})$ . Since this subgroup has finite index,  $\Lambda$  and  $\bar{p}_k$  agree on the whole group. □

**5.2. Proof of Theorem 1 (b).** We show first that if  $f: S^{4k-1} \rightarrow \mathbb{R}^{4k+1}$  is a generic immersion then the expression

$$\Theta(f) = -\bar{p}_k[\hat{M}^{4k}] + 3t(g) - 3l(g) + L(\partial g), \quad (9)$$

is independent of the choice of the map  $g: M^{4k} \rightarrow \mathbb{R}_+^{6k}$  and invariant under regular homotopy of  $f$ .

Let  $g_0: M_0^{4k} \rightarrow \mathbb{R}_+^{6k}$  and  $g_1: M_1^{4k} \rightarrow \mathbb{R}_+^{6k}$  be two generic maps with  $\partial g_0$  and  $\partial g_1$  both regularly homotopic to  $j \circ f$ .

Then  $\partial g_0$  is regularly homotopic to  $\partial g_1$ . Let  $k_t$  be a generic regular homotopy between them and let  $K: S^{4k-1} \times I \rightarrow \mathbb{R}^{6k-1} \times I$  be the map  $K(x, t) = (k_t(x), t)$ . To each triple point instance of  $k_t$  there corresponds a change in  $L(k_t)$  by  $\pm 3$  (see [4] or [6]), and also a triple point of  $K$  which has the same sign as the sign of the change in  $L(k_t)$ . Thus,

$$L(\partial g_1) - L(\partial g_0) = 3t(K). \quad (10)$$

Let  $-g_0: -M_0^{4k} \rightarrow \mathbb{R}_-^{6k}$  be the map  $r \circ g_0$ , where  $r: \mathbb{R}^{6k} \rightarrow \mathbb{R}^{6k}$  is the reflection in  $\partial \mathbb{R}_+^{6k}$ . Using  $K$  to glue  $-g_0$  and a vertically translated copy of  $g_1$ , we get a map

$h: -\hat{M}_0^{4k} \# \hat{M}_1^{4k} \rightarrow \mathbb{R}^{6k}$ . Then, by Lemma 4,

$$\begin{aligned} 0 &= -\bar{p}_k[-\hat{M}_0^{4k} \# \hat{M}_1^{4k}] + 3t(h) - 3l(h) \\ &= -\bar{p}_k[\hat{M}_1^{4k}] + \bar{p}_k[\hat{M}_0^{4k}] + 3(t(g_1) - t(g_0) + t(K)) - 3(l(g_1) - l(g_0)). \end{aligned}$$

Together with Equation (10), this implies that  $\Theta(f)$  is independent of the choice of  $g$  and also that  $\Theta$  is invariant under regular homotopy. (Actually, it implies something stronger:  $\Theta$  only depends on the regular homotopy class of  $j \circ f$  in  $\mathbb{R}^{6k-1}$ . This observation leads to Corollary 1.)

Clearly,  $\Theta$  is additive under connected sum and hence induces a homomorphism  $\Theta: \text{Imm}(S^{4k-1}, \mathbb{R}^{4k+1}) \rightarrow \mathbb{Z}$ . Thus,  $\Theta = c \cdot \Omega$ , where  $c \in \mathbb{Z}$  is some constant and  $\Omega$  is the Smale invariant.

Then according to Lemma 1,  $c = a_k(2k-1)!$ . Indeed, let  $g: M^{4k} \rightarrow \mathbb{R}^{4k+1} \subset \mathbb{R}^{6k-1}$  be a Seifert-surface of an embedding  $\partial g: S^{4k-1} \rightarrow \mathbb{R}^{4k+1}$ . Push the interior of  $g(M^{4k})$  into  $\mathbb{R}_+^{6k}$ . Then  $l(g) = 0$  and  $L(\partial g) = 0$  since  $g$  has neither singularities nor self intersections. Hence,  $\Theta(f) = -\bar{p}_k[\hat{M}^{4k}]$  and the latter is  $a_k(2k-1)! \cdot \Omega(f)$  by Lemma 1.  $\square$

## 6. GENERIC $4k$ -MANIFOLDS IN $(6k-1)$ -SPACE AND THE PROOF OF THEOREM 1 (a)

In this section we show that the right-hand sides in (1) and (2) agree and demonstrate, after introducing a small amount of cobordism theory, how Theorem 1 (a) follows from Theorem 1 (b).

**6.1. A vector bundle over singularities.** Let  $g: M^{4k} \rightarrow \mathbb{R}^{6k-1}$  be a generic map of a compact orientable manifold (see Remark 2). Then the singularity set  $\tilde{\Sigma}(g)$  of the map  $g: M^{4k} \rightarrow \mathbb{R}^{6k-1}$  is a  $2k$ -dimensional submanifold of  $M^{4k}$ , which is not necessarily orientable, see Example 1. Recall, see Definition 8, that  $\xi(g)$  is the  $2k$ -dimensional vector bundle over  $\tilde{\Sigma}(g)$ , the fiber of which at a point  $p \in \tilde{\Sigma}(g)$  is  $T_{g(p)}\mathbb{R}^{6k-1}/dg(T_p M^{4k})$ .

**Lemma 5.** *The three line bundles  $\det(T\tilde{\Sigma}(g))$ ,  $\det(\xi(g))$ , and  $\ker(dg)$  over  $\tilde{\Sigma}(g)$  are isomorphic. The total space  $E(g)$  of the vector bundle  $\xi(g)$  is orientable. Moreover, orientations on  $M^{4k}$  and  $\mathbb{R}^{6k-1}$  induce an orientation on  $E(g)$ .*

*Proof.* If  $p$  is a point in  $\tilde{\Sigma}(g)$  then,

$$T_{g(p)}\mathbb{R}^{6k-1} = dg(T_p M^{4k} / \ker(dg)_p) \oplus \xi(g)_p.$$

Hence, over  $\tilde{\Sigma}(g)$ , we have the following isomorphism of line bundles:

$$\det(g^*(T\mathbb{R}^{6k-1})) = \det(TM^{4k}) \otimes \ker(dg) \otimes \det(\xi(g)).$$

It follows that orientations on  $M^{4k}$  and  $\mathbb{R}^{6k}$  induce an isomorphism

$$\det(\xi(g)) \approx \ker(dg). \quad (11)$$

Consider a point  $p \in \tilde{\Sigma}(g)$  which is not a cusp. Let  $U$  be a small neighborhood of  $p$  in  $M^{4k}$  in which there are no cusps. Let  $g' = g|_U$ . If  $U$  is chosen small enough then  $g'$  does not have any triple points,  $\tilde{\Sigma}(g') = \tilde{\Sigma}(g) \cap U$ ,  $X(g') = \tilde{D}_2(g') \cup \tilde{\Sigma}(g')$  is an embedded submanifold of  $U$ , and  $\tilde{\Sigma}(g')$  divides  $X(g')$  into two connected components  $\tilde{D}_2^+(g')$  and  $\tilde{D}_2^-(g')$ .

Let  $x \in \tilde{D}_2(g')$ . Then there exists a unique  $y \in \tilde{D}_2(g')$  such that  $g'(x) = g'(y)$ . Moreover, there are neighborhoods  $V$  of  $x$  and  $W$  of  $y$  in  $U$  such that  $A(x) =$

$g'(V) \cap g'(W) \subset \mathbb{R}^{6k-1}$  is canonically diffeomorphic to a neighborhood  $B(x)$  of  $x$  in  $X(g')$ . We pull back the orientation induced on  $A(x)$  by considering it as the *ordered* intersection  $g'(V) \cap g'(W)$  to  $B(x)$ . In this way we get an orientation of  $X(g') - \tilde{\Sigma}(g)$ .

Note that the orientation on  $A = A(x) = A(y)$  pulled back to the neighborhood  $B(x) \subset X(g')$  of  $x$  is *opposite* to that pulled back to the neighborhood  $B(y) \subset X(g')$  of  $y$ . Thus, the orientations induced on  $\tilde{\Sigma}(g')$  as the boundary of  $\tilde{D}_2^+(g')$  and  $\tilde{D}_2^-(g')$ , respectively are *opposite*. This means that the orientation of  $\tilde{D}_2(g')$  extends over  $\tilde{\Sigma}(g')$  and gives an orientation of  $X(g')$ .

Along  $\tilde{\Sigma}(g')$ , we have  $TX(g')|_{\tilde{\Sigma}(g')} = T\tilde{\Sigma}(g') \oplus \ker(dg')$ . This is a subbundle of  $TM^{4k}|_{\tilde{\Sigma}(g')}$ . Let  $l$  be a  $(2k-1)$ -dimensional subbundle of  $TM^{4k}|_{\tilde{\Sigma}(g')}$  complementary to  $TX(g')|_{\tilde{\Sigma}(g')}$ , i.e. such that, there is a bundle isomorphism

$$TM^{4k}|_{\tilde{\Sigma}(g')} = TX(g')|_{\tilde{\Sigma}(g')} \oplus l.$$

Then orientations of  $X(g')$  and  $TM^{4k}$  induce an orientation on  $l$ . The decomposition

$$TM^{4k}|_{\tilde{\Sigma}(g')} = T\tilde{\Sigma}(g') \oplus \ker(dg') \oplus l,$$

gives the line bundle isomorphism

$$\det(TM^{4k}|_{\tilde{\Sigma}(g')}) = \det(T\tilde{\Sigma}(g')) \otimes \ker(dg') \otimes \det(l).$$

The orientations on  $M^{4k}$  and  $l$  then induce an isomorphism

$$\det(T\tilde{\Sigma}(g')) \approx \ker(dg').$$

Now  $T_p\tilde{\Sigma}(g') = T_p\tilde{\Sigma}(g)$  and  $\ker(dg')_p = \ker(dg)_p$ . Hence, we have an isomorphism

$$\det(T\tilde{\Sigma}(g)) \approx \ker(dg), \quad (12)$$

over  $\tilde{\Sigma}(g) - C$ , where  $C$  denotes the set of cusp points. Since  $C$  has codimension  $2k$  in  $\tilde{\Sigma}(g)$  the isomorphism extends uniquely over  $C$ .

Finally, if  $v$  is a point in  $E(g)$ , with  $\pi(v) = p$ , where  $\pi: E(g) \rightarrow \tilde{\Sigma}(g)$  is the projection. Then the tangent space of  $E(g)$  at  $v$  splits as

$$T_v E(g) \approx T_p \tilde{\Sigma}(g) \oplus \xi(g)_p.$$

Hence,

$$\det(TE(g)) \approx \ker(dg) \otimes \ker(dg),$$

is trivial and  $E(g)$  is orientable. Moreover, the isomorphisms (11) and (12) give a trivialization of  $\det(TE(g))$ . That is, there is an induced orientation on  $E(g)$ .  $\square$

**Remark 13.** Alternatively, one can prove Lemma 5 by using the diagram

$$\begin{array}{ccc} M^{4k} \times M^{4k} & \xrightarrow{g \times g} & \mathbb{R}^{6k-1} \times \mathbb{R}^{6k-1} \\ \phi \downarrow & & \downarrow \phi \\ M^{4k} \times M^{4k} & \xrightarrow{g \times g} & \mathbb{R}^{6k-1} \times \mathbb{R}^{6k-1} \end{array},$$

where  $\phi(x, y) = (y, x)$ , to show that the closure  $X(g)$  of the double point manifold of  $g$  in  $M^{4k}$  is oriented. With this shown, one can use the identification of the normal bundle of  $\tilde{\Sigma}(g) - C$  ( $C$  is the set of cusp points) in  $X(g)$  with  $\ker(dg)$  to establish the isomorphism  $\det(T\tilde{\Sigma}(g)) \approx \ker(dg)$ .

Lemma 5 says that  $\xi(g)$  is a  $2k$ -dimensional vector bundle over the  $2k$ -dimensional manifold  $\tilde{\Sigma}(g)$  with total space  $E(g)$  which is oriented. There is an Euler number associated to such a bundle. In terms of homology with local coefficients this Euler number is derived as follows:

Let  $\mathcal{K}$  be the twisted integer local coefficient system associated to the orientation bundle of  $\tilde{\Sigma}$  and let  $\mathcal{F}$  denote the local system associated to the fiber orientation of the bundle  $\xi(g)$ . We have proved above that  $\mathcal{K} \approx \mathcal{F}$  (both are isomorphic to the sheaf of unit length sections of  $\ker(dg)$ ). The orientation on the total space gives the relation

$$\mathcal{K} \otimes \mathcal{F} = \mathbb{Z}, \quad (13)$$

where  $\mathbb{Z}$  is the (trivialized) local coefficient system associated to the orientation bundle of  $E(g)$  restricted to the zero-section. This relation is used to specify an isomorphism  $\mathcal{K} \approx \mathcal{F}$ , by requiring that at each point it is given by ordinary multiplication of integers. This specified isomorphism gives in turn a well-defined pairing

$$H^{2k}(\tilde{\Sigma}(g); \mathcal{F}) \otimes H_{2k}(\tilde{\Sigma}(g), \mathcal{K}) \rightarrow \mathbb{Z}.$$

Especially, we get the Euler number  $e(\xi(g)) = \langle e, [\tilde{\Sigma}(g)] \rangle$ , where  $e \in H^{2k}(\tilde{\Sigma}(g); \mathcal{F})$  is the Euler class of  $\xi(g)$  and  $[\tilde{\Sigma}(g)] \in H_{2k}(\tilde{\Sigma}(g), \mathcal{K})$  is the orientation class of  $\tilde{\Sigma}(g)$ .

One can compute the above Euler number by choosing a section  $s$  of  $\xi(g)$  transverse to the zero-section and sum up the local intersection numbers at the zeros of  $s$ . The local intersection number at a zero  $p$  of  $s$  is the intersection number in  $E(g)$  of the zero-section with some local orientation on a neighborhood  $U$  of  $p$  and  $s(U)$  with the orientation induced from that chosen local orientation on  $U$ .

## 6.2. Prim maps.

**Definition 12.** Let  $M^n$  be a manifold. A map  $g: M^n \rightarrow \mathbb{R}^{n+m}$  is a prim map (projected immersion) if there exists an immersion  $G: M^n \rightarrow \mathbb{R}^{n+m+1}$  such that  $g = \pi \circ G$ , where  $\pi: \mathbb{R}^{n+m+1} \rightarrow \mathbb{R}^{n+m}$  is the projection forgetting the last coordinate.

**Remark 14.** The bundle  $\xi = \xi(g)$  has a more simple description if  $g: M^{4k} \rightarrow \mathbb{R}^{6k-1}$  is a prim map. Assume that  $g = \pi \circ G$ . Then  $\xi$  is isomorphic to the normal bundle  $\nu$  of  $G$  restricted to  $\tilde{\Sigma}(g)$ .

Since the homology class of  $\tilde{\Sigma}(g)$  in  $M^{4k}$  is dual to the normal Euler class of the immersion  $G$ , we have (if  $e(\eta)$  denotes the Euler class of the bundle  $\eta$ )

$$\langle e(\xi), [\tilde{\Sigma}] \rangle = \langle e^2(\nu), [M^{4k}] \rangle = \bar{p}_k[M^{4k}].$$

**6.3. The Euler class and cusps.** In this section we prove that the expressions in Equations (1) and (2) agree. We use the notation of Section 2.4.

**Lemma 6.** If  $g: M^{4k} \rightarrow \mathbb{R}^{6k-1}$  is a generic map of a closed oriented manifold then

$$\bar{p}_k(M^{4k}) = \sharp^{\Sigma^{1,1}}(g).$$

*Proof.* This is Lemma 3 in [25]. □

**Lemma 7.** Let  $M^{4k}$  be a closed oriented manifold and let  $g: M^{4k} \rightarrow \mathbb{R}^{6k-1}$  be a generic map. Then

$$e(\xi(g)) = \sharp^{\Sigma^{1,1}}(g). \quad (14)$$

*Proof.* If  $g$  is a prim map then both sides in Equation (14) equal  $\bar{p}_k[M^{4k}]$ , see Remark 14 for the left-hand side and Lemma 6 for the right-hand side. Both sides are invariant under cobordisms of generic maps and hence they define homomorphisms  $\Omega_{4k}(\mathbb{R}^{6k-1}) \rightarrow \mathbb{Z}$ . By Burlet's theorem [2] (see the proof of Lemma 4) the classes representable by prim maps form a subgroup of finite index. Since the homomorphisms agree on this subgroup they agree on the whole group.  $\square$

**Remark 15.** Alternatively, Lemma 7 may be proved as follows:

The second derivative  $D^2g$  (intrinsic derivative of Porteous) of the map  $g$  is a quadratic form on  $\ker(dg)$  with values in  $\xi(g) = T\mathbb{R}^{6k-1}/\text{im}(dg)$ . Choose a Riemannian metric on  $\ker(dg)$  and let  $\pm v(x)$  denote the unit vectors in each fiber over  $x \in \Sigma(g)$ . Then  $s(x) = D^2g(\pm v(x), \pm v(x))$  gives a section of  $\xi(g)$ . This section vanishes exactly at the  $\Sigma^{1,1}$ -points. (Confer Section 2.2 of [10] which deals with the cusp-free case.) The Euler number of the bundle  $\xi(g)$  is the sum of local intersection numbers at the zeros of  $s$ .

**6.4. Cobordism groups of maps and natural homomorphisms.** We shall consider classifying spaces of certain maps of codimension  $m$  into Euclidean space.

**Definition 13.** (a) Let  $X(m)$  be the classifying space of (generic) codimension  $m$  maps of closed oriented manifolds into Euclidean space.

(b) Let  $\bar{X}(m)$  be the classifying space of codimension  $m$  prim maps of closed oriented manifolds into Euclidean space.

(c) Let  $\Gamma(m)$  be the classifying space of codimension  $m$  immersions of closed oriented manifolds into Euclidean space.

(d) Let  $\Gamma_{\text{fr}}(m)$  be the classifying space of codimension  $m$  framed immersions of closed oriented manifolds into Euclidean space.

Definition 13 (a) means that  $\pi_{n+m}(X(m))$  is the cobordism group of arbitrary maps of oriented  $n$ -manifolds in  $\mathbb{R}^{n+m}$ . Definition 13 (b)-(d) give similar interpretations of the homotopy groups of the corresponding spaces.

**Remark 16.** Up to homotopy equivalence these spaces can be identified as follows:

(a)

$$X(m) = \lim_{K \rightarrow \infty} \Omega^K MSO(K+m),$$

(b)

$$\bar{X}(m) = \lim_{K \rightarrow \infty} \Omega^{K+1} S^K MSO(m+1),$$

(c)

$$\Gamma(m) = \lim_{K \rightarrow \infty} \Omega^K S^K MSO(m),$$

(d)

$$\Gamma_{\text{fr}}(m) = \lim_{K \rightarrow \infty} \Omega^K S^{K+m},$$

where  $\Omega^j$  denotes the  $j^{\text{th}}$  loop space and  $S^j$  the  $j^{\text{th}}$  suspension or the  $j$ -dimensional sphere.

We note that there are natural inclusions among these spaces and that the corresponding relative homotopy groups also have concrete geometric interpretations. For example,  $\pi_{n+m}(\bar{X}(m), \Gamma_{\text{fr}}(m))$  is the cobordism group of prim maps  $(M^n, \partial M^n) \rightarrow (\mathbb{R}_+^{n+m}, \partial \mathbb{R}_+^{n+m})$  which are framed immersions on the boundary.

**Definition 14.** Let  $\beta$  be an element of  $\pi_{6k-1}(\bar{X}(2k-1))$  or of  $\pi_{6k-1}(\bar{X}(2k-1), \Gamma_{\text{fr}}(2k-1))$ , or of  $\pi_{6k-1}(X(2k-1), \Gamma_{\text{fr}}(2k-1))$ . Represent  $\beta$  by a generic map  $g: M^{4k} \rightarrow \mathbb{R}^{6k-1}$  and define

$$\Sigma^{1,1}(\beta) = \sharp \Sigma^{1,1}(g),$$

see Definition 7.

Note that a generic cobordism between two generic maps as in Definition 14 gives a cobordism between their 0-dimensional manifolds of cusp points. Hence, the function  $\Sigma^{1,1}$  is a well defined homomorphism.

**Definition 15.** Let  $\beta$  be an element of  $\pi_{6k}(\Gamma(2k))$ . Represent  $\beta$  by a generic (self transverse) immersion  $g: M^{4k} \rightarrow \mathbb{R}^{6k}$  and define

$$3t(\beta) = 3t(g),$$

see Definition 5.

As above we see that  $3t$  is a well defined homomorphism.

**Definition 16.** Let  $\beta$  be an element of  $\pi_{6k}(\Gamma(2k), \Gamma_{\text{fr}}(2k))$ . Represent  $\beta$  by a generic (self transverse) immersion  $g: (M^{4k}, \partial M^{4k}) \rightarrow (\mathbb{R}_+^{6k}, \partial \mathbb{R}_+^{6k})$  and define

$$\Phi(\beta) = 3t(g) + L(\partial g),$$

see Definitions 5 and 4.

**Definition 17.** Let  $\beta$  be an element of  $\pi_{6k}(X(2k), \Gamma_{\text{fr}}(2k))$ . Represent  $\beta$  by a generic map  $g: (M^{4k}, \partial M^{4k}) \rightarrow (\mathbb{R}_+^{6k}, \partial \mathbb{R}_+^{6k})$  and define

$$\Psi(\beta) = 3t(g) - 3l(g) + L(\partial g),$$

see Definitions 5, 4, and 6.

By the same argument as is used in the proof of Theorem 1 (b) to show that  $\Theta$  (see Equation (9)) is well defined, it follows that  $\Phi$  and  $\Psi$  as defined above are well defined homomorphisms.

The following lemma is the main step in the proof of Theorem 1 (a).

**Lemma 8.** Let

$$i: \pi_{6k-1}(X(2k-1), \Gamma_{\text{fr}}(2k-1)) \rightarrow \pi_{6k}(X(2k), \Gamma_{\text{fr}}(2k))$$

be the natural homomorphism. Then  $\Sigma^{1,1} = \Psi \circ i$ .

*Proof.* To shorten the notation, let  $m = 2k-1$  and  $n = 2k$ . Consider the following diagram:

$$\begin{array}{ccccc} \pi_{6k-1}(\bar{X}(m)) & \longrightarrow & \pi_{6k-1}(\bar{X}(m), \Gamma_{\text{fr}}(m)) & \longrightarrow & \pi_{6k-1}(X(m), \Gamma_{\text{fr}}(m)) \\ i'' \downarrow & & i' \downarrow & & \downarrow i \\ \pi_{6k}(\Gamma(n)) & \longrightarrow & \pi_{6k}(\Gamma(n), \Gamma_{\text{fr}}(n)) & \longrightarrow & \pi_{6k}(X(n), \Gamma_{\text{fr}}(n)) \end{array}$$

The horizontal homomorphisms are obtained by forgetting structure. The vertical homomorphisms are the natural ones induced by the (framed) inclusion  $\mathbb{R}^{6k-1} \rightarrow \mathbb{R}^{6k}$ . The diagram clearly commutes.

We now show that the horizontal homomorphisms in these sequences all have finite cokernels. We start with the first horizontal arrow. This homomorphism is

part of the long exact homotopy sequence of the pair  $(\bar{X}(m), \Gamma_{\text{fr}}(m))$ . Consider the following fragment of this sequence:

$$\dots \pi_{6k-1}(\bar{X}(m)) \longrightarrow \pi_{6k-1}(\bar{X}(m), \Gamma_{\text{fr}}(m)) \longrightarrow \pi_{6k-2}(\Gamma_{\text{fr}}(m)) \dots$$

The group  $\pi_{6k-2}(\Gamma_{\text{fr}}(m))$  is the stable homotopy group of spheres  $\pi^s(4k-1)$ , which is finite. It follows that the cokernel is finite.

Similarly, if the groups

- (a)  $\pi_{6k-1}(X(m), \bar{X}(m))$ ,
- (b)  $\pi_{6k-1}(\Gamma_{\text{fr}}(n))$ , and
- (c)  $\pi_{6k}(X(n), \Gamma(n))$ ,

are finite then the other horizontal homomorphisms have finite cokernels.

The group in (b) is again a stable homotopy group of spheres and therefore finite. The group in (c) is finite by Burlet's theorem [2] (see the proof of Lemma 4). Finally, the group in (a) is isomorphic to the group in (c):

Any map  $f: (M^{4k}, \partial M^{4k}) \rightarrow (\mathbb{R}_+^{6k-1}, \partial \mathbb{R}_+^{6k-1})$  which is prim on the boundary lifts to a map  $F: (M^{4k}, \partial M^{4k}) \rightarrow (\mathbb{R}_+^{6k}, \partial \mathbb{R}_+^{6k})$  which is an immersion on the boundary. This gives the isomorphism  $\pi_{6k-1}(X(m), \bar{X}(m)) \rightarrow \pi_{6k}(X(n), \Gamma(n))$  on representatives. The inverse is induced by the projection  $(\mathbb{R}_+^{6k}, \partial \mathbb{R}_+^{6k}) \rightarrow (\mathbb{R}_+^{6k-1}, \partial \mathbb{R}_+^{6k-1})$ .

By Remark 1 in [25], it follows that  $\Sigma^{1,1} = 3t \circ i''$ . Since all cokernels are finite, we conclude first that  $\Sigma^{1,1} = \Phi \circ i'$  and then that  $\Sigma^{1,1} = \Psi \circ i$ .  $\square$

**6.5. Proof of Theorem 1 (a).** Let  $f: S^{4k-1} \rightarrow \mathbb{R}^{4k+1}$  be an immersion. Then there is a homotopically unique normal framing of  $f$ . Thus, the immersion  $j \circ f: S^{4k-1} \rightarrow \mathbb{R}^{6k-1}$  is also framed.

Let  $g: M^{4k} \rightarrow \mathbb{R}^{6k-1}$  be a generic map of a manifold with spherical boundary such that  $\partial g$  is regularly homotopic to  $j \circ f$ . Then there is an induced normal framing of  $\partial g$ . After composing  $g$  with a translation, we can assume that  $g(M^{4k})$  is contained in the half space on which the last coordinate function is strictly positive.

Applying Hirsch lemma to any vector field in the normal framing of  $\partial g$ , we find a homotopy of  $M^{4k}$  supported in a small collar of the boundary  $\partial M^{4k}$ , which is a regular homotopy when restricted to this collar and a framed regular homotopy of  $\partial M^{4k}$  and which deforms  $\partial g$  to an immersion mapping into  $\partial \mathbb{R}_+^{6k-1}$ .

Let  $g': (M^{4k}, \partial M^{4k}) \rightarrow (\mathbb{R}_+^{6k-1}, \partial \mathbb{R}_+^{6k-1})$  denote the map obtained from  $g$ . Then  $g'$  represents an element  $\zeta \in \pi_{6k-1}(X(2k-1), \Gamma_{\text{fr}}(2k-1))$ .

Theorem 1 (b) says that  $a_k(2k-1)! \cdot \Omega(f) + \bar{p}_k[\hat{M}^{4k}] = \Psi(i(\zeta))$ . Thus, by Lemma 8, we have

$$a_k(2k-1)! \cdot \Omega(f) + \bar{p}_k[\hat{M}^{4k}] = \Sigma^{1,1}(\zeta) = \sharp \Sigma^{1,1}(g') = \sharp \Sigma^{1,1}(g).$$

This proves Equation (1). Equation (2) then follows from Lemma 7.  $\square$

## 7. $\text{Imm}(S^{4k-1}, \mathbb{R}^{4k+1}) \rightarrow \text{Imm}(S^{4k-1}, \mathbb{R}^{6k-1})$ IS INJECTIVE

In this section we prove Corollary 1.

**7.1. Proof of Corollary 1.** Let  $f, g: S^{4k-1} \rightarrow \mathbb{R}^{4k+1}$  be immersions such that  $j \circ f: S^{4k-1} \rightarrow \mathbb{R}^{6k-1}$  is regularly homotopic to  $j \circ g$ . Theorem 1 applied to a generic immersion  $\partial h: S^{4k-1} \rightarrow \mathbb{R}^{6k-1}$ , in the common regular homotopy class of  $j \circ f$  and  $j \circ g$ , bounding a generic map  $h: M^{4k} \rightarrow \mathbb{R}_+^{6k}$ , shows that  $\Omega(f) = \Omega(g)$ . In other words,  $\text{Imm}(S^{4k-1}, \mathbb{R}^{4k+1}) \rightarrow \text{Imm}(S^{4k-1}, \mathbb{R}^{6k-1})$  is injective.

Let  $f, g: S^{4k-1} \rightarrow \mathbb{R}^{4k+1}$  be immersions such that  $\Omega(f) - \Omega(g) = c$ . Let  $h: M^{4k} \rightarrow \mathbb{R}^{6k-1}$  be a generic map such that  $\partial h = j \circ g$ . If  $F: S^{4k-1} \times I \rightarrow \mathbb{R}^{6k-1}$  is a generic homotopy connecting  $j \circ f$  to  $j \circ g$  then we glue the collar  $S^{4k-1} \times I$  to  $M^{4k}$  and obtain a manifold  $N^{4k}$  with a generic map  $k: N^{4k} \rightarrow \mathbb{R}^{6k-1}$ , which equals  $F$  on the collar and  $h$  on  $M^{4k}$ . Now,  $\partial k = j \circ f$  and  $\hat{M}^{4k} \approx \hat{N}^{4k}$ . Hence, by Theorem 1 (a)

$$c \cdot a_k(2k-1)! = (\Omega(f) - \Omega(g)) \cdot a_k(2k-1)! = \sharp^{\Sigma^{1,1}}(k) - \sharp^{\Sigma^{1,1}}(h) = \sharp^{\Sigma^{1,1}}(F).$$

□

## 8. ON THE NUMBER OF $(2k+1)$ -TUPLE POINTS OF AN IMMERSION $M^{4k} \rightarrow R_+^{4k+2}$ BOUNDING A GIVEN IMMERSION

In this section we prove Theorem 2.

**8.1. Proof of Theorem 2.** Let  $\Theta(f)$  denote the left-hand side of Equation (4). We shall show that

- (a)  $\Theta(f)$  is well-defined, i.e. it does not depend on the choices of  $d$  and  $g$ .
- (b)  $\Theta(f)$  does not change if the immersion  $f$  changes in its cobordism class.

But first we show that the theorem follows from (a) and (b):

Let  $\mathbf{Imm}^{SO}(4k-1, 2)$  be the cobordism group of immersions of oriented  $(4k-1)$ -manifolds in  $\mathbb{R}^{4k+3}$ . Then  $\mathbf{Imm}^{SO}(4k-1, 2)$  is isomorphic to  $\pi_{4k+3}^s(\mathbb{C}P^\infty)$ , the  $(4k+3)^{\text{th}}$  stable homotopy group of  $\mathbb{C}P^\infty$ . By Serre's theorem,  $\pi_n^s(X) \otimes \mathbb{Q} \approx H_n(X) \otimes \mathbb{Q}$ , and hence,  $\mathbf{Imm}^{SO}(4k-1, 2)$  is finite.

Let  $\mathcal{I}_2$  be the subgroup of  $\mathbf{Imm}^{SO}(4k-1, 2)$  which consists of cobordism classes representable by immersions of 2-connected manifolds. By (a) and (b), the formula  $\Theta(f)$  defines a homomorphism  $\Theta: \mathcal{I}_2 \rightarrow \mathbb{Z}$ . Since  $\mathcal{I}_2$  is finite, this homomorphism must be identically zero.

We now return to the proofs of (a) and (b).

The statement (a) follows from a special case of Herbert's theorem [11] saying that if  $h: N^{4k} \rightarrow \mathbb{R}^{4k+2}$  is an immersion of a *closed* manifold then  $\langle \bar{p}_1^k, [N^{4k}] \rangle = \sharp D_{2k+1}(h)$ .

Consider (b). Let  $F: W^{4k} \rightarrow \mathbb{R}^{4k+1} \times I$  be a generic immersion, which is a cobordism between  $f_0: V_0 \rightarrow \mathbb{R}^{4k+1} \times 0$  and  $f_1: V_1 \rightarrow \mathbb{R}^{4k+1} \times 1$ . We must show.

$$L_{2k}(f_1) - L_{2k}(f_0) = \bar{p}_1^k[W^{4k}] + (2k+1)\sharp D_{2k+1}(F).$$

We first introduce some notation: Let  $\Delta_{2k}(F)$  denote the resolved  $2k$ -fold self intersection manifold of  $F$ . (Then there is a map  $\Delta_{2k}(F) \rightarrow D_{2k}(F) \cup D_{2k+1}(F)$ , which is a diffeomorphism when restricted to the preimage of  $D_{2k}(F)$  and in the preimage of each point in  $D_{2k+1}(F)$  there are exactly  $2k+1$  points.) Similarly, let  $\tilde{\Delta}_{2k}(F)$  denote the resolution of  $\tilde{D}_{2k}(F) \cup \tilde{D}_{2k+1}(F)$ . Then there is a  $2k$ -fold covering  $\pi: \tilde{\Delta}_{2k}(F) \rightarrow \Delta_{2k}(F)$ . Also, let  $j: \tilde{\Delta}_{2k}(F) \rightarrow W^{4k}$  and  $i: \Delta_{2k}(F) \rightarrow \mathbb{R}^{4k+2} \times I$  denote the obvious immersions.

Let  $\nu_F$  denote the normal bundle of  $F$  and let  $\zeta = j^*(\nu_F)$ . The 2-connectedness of  $V_0$  and  $V_1$  implies that  $\nu_F|_{\partial W^{4k}}$  has a homotopically unique trivialization. The same is then true for the restriction of  $\zeta$  to the boundary. Let  $s$  be a section of  $\zeta$  which does not vanish on the boundary.

The normal bundle of  $i: \Delta_{2k}(F) \rightarrow \mathbb{R}^{4k+1} \times I$  is then  $\pi_1(\zeta)$  and the section  $s$  gives a section  $z$  of  $\pi_1(\zeta)$ , namely  $z(q) = s(q_1) + \cdots + s(q_{2k})$ , where  $q = \pi(q_1) = \cdots = \pi(q_{2k})$ .

Let  $i': \Delta_{2k}(F) \rightarrow \mathbb{R}^{4k+2} \times I$  be given by  $i$  shifted a small distance along  $z$  (i.e.  $i'(x) = i(x) + \epsilon z(x)$ , where  $\epsilon > 0$  is very small). Then  $i'(\Delta_{2k}(F)) \cap F(W^{4k})$  is a collection of points. The points are of two types:

- (i) Near each  $(2k+1)$ -fold self intersection point  $p$  of  $F$  there are  $2k+1$  intersection points, all with the same local intersection number, which is the sign of the  $(2k+1)$ -fold self intersection point  $p$ .
- (ii) One intersection point for each zero of  $s$ . There is a local intersection number associated to such a point.

Clearly, we can choose  $s$  so that the sets of intersection points of type (i) and (ii), respectively, are disjoint.

Hence,

$$i'(\Delta_{2k}(F)) \bullet F(W^{4k}) = (2k+1)\sharp D_{2k+1}(F) + \sharp\{s^{-1}(0)\},$$

where  $\sharp\{s^{-1}(0)\}$  denotes the algebraic number of intersection points of type (ii). Applying Lemma 2, we find that

$$L_{2k}(f_1) - L_{2k}(f_0) = (2k+1)\sharp D_{2k+1}(F) + \sharp\{s^{-1}(0)\}.$$

Now,  $\sharp\{s^{-1}(0)\}$  equals the relative Euler class  $e(\zeta) \in H^2(\tilde{\Delta}_{2k}(F), \partial\tilde{\Delta}_{2k}(F))$  of the bundle  $\zeta$ . The immersed submanifold  $j(\tilde{\Delta}_{2k}(F))$  represents the class dual to  $(2k-1)^{\text{th}}$  power of the Euler class of  $\nu(F)$ . If  $\mathcal{D}$  is the Poincaré duality operator on  $W = W^{4k}$  then

$$\begin{aligned} \sharp\{s^{-1}(0)\} &= \left\langle e(j^*\nu_F), [\tilde{\Delta}_{2k}(F), \partial\tilde{\Delta}_{2k}(F)] \right\rangle = \left\langle j^*e(\nu_F), [\tilde{\Delta}_{2k}(F), \partial\tilde{\Delta}_{2k}(F)] \right\rangle = \\ &= \left\langle e(\nu_F), j_*[\tilde{\Delta}_{2k}(F), \partial\tilde{\Delta}_{2k}(F)] \right\rangle = \left\langle e(\nu_F), \mathcal{D}e^{2k-1}(\nu(F)) \right\rangle = \\ &= \langle e^{2k}(\nu_F), [W, \partial W] \rangle = \langle \bar{p}_1^k, [W, \partial W] \rangle. \end{aligned}$$

□

## 9. CODIMENSION TWO IMMERSIONS OF SPHERES OF DIMENSIONS $8k+5$ AND $8k+1$

In this section we state a formula which might give the Smale invariant of an immersion  $S^{8k+1} \rightarrow \mathbb{R}^{8k+3}$  and prove that the corresponding formula vanishes identically for immersions  $S^{8k+5} \rightarrow \mathbb{R}^{8k+7}$ .

**9.1. A brief discussion of definitions and notation.** Let  $f: S^{8k+1} \rightarrow \mathbb{R}^{8k+3}$  ( $f: S^{8k+5} \rightarrow \mathbb{R}^{8k+7}$ ) be an immersion. Let  $j: \mathbb{R}^{8k+3} \rightarrow \mathbb{R}^{12k+2}$  ( $j: \mathbb{R}^{8k+7} \rightarrow \mathbb{R}^{12k+8}$ ) be the inclusion. Let  $g: M^{8k+2} \rightarrow \mathbb{R}_+^{12k+3}$  ( $g: M^{8k+6} \rightarrow \mathbb{R}_+^{12k+9}$ ) be a generic map of a compact manifold such that  $\partial g$  is an immersion regularly homotopic to  $j \circ f$ .

The  $\mathbb{Z}_2$ -valued invariants  $t(g)$ ,  $l(g)$ , and  $L(\partial g)$  are then defined as the corresponding invariants in Definitions 5, 6, and 4, respectively, with the following modifications: First,  $\mathbb{Z}_2$  is used instead of  $\mathbb{Z}$  and second, no orientations are needed.

For a compact closed  $2j$ -dimensional manifold  $M^{2j}$ , let  $\bar{w}_j$  denote the  $j^{\text{th}}$  normal Stiefel-Whitney class of  $M^{2j}$  and let  $\bar{w}_j^2[M^{2j}] \in \mathbb{Z}_2$  denote the corresponding Stiefel-Whitney number.

**9.2. Possibly a Smale invariant formula.** Let  $f: S^{8k+1} \rightarrow \mathbb{R}^{8k+3}$  be an immersion. Let  $\Omega(f) \in \mathbb{Z}_2$  be its Smale invariant and let  $g$  be as in Section 9.1.

**Question 1.** *Either*

$$\Omega(f) = w_{4k+1}^2[\hat{M}^{8k+2}] + t(g) + l(g) + L(\partial g), \quad (15)$$

*or*

$$w_{4k+1}^2[\hat{M}^{8k+2}] + t(g) + l(g) + L(\partial g) = 0, \quad (16)$$

where we use the notations introduced in Section 9.1. Which one is true? (Note that Equations (15) and (16) are equations in  $\mathbb{Z}_2$ .)

Question 1 can be treated in the same way as Theorem 1 (b). In the proof of Theorem 1 (b) we used Lemma 4. The analog of Lemma 4 in the present situation is the following:

**Lemma 9.** *Let  $M^{2n}$  be a closed  $2n$ -dimensional manifold and let  $h: M^{2n} \rightarrow \mathbb{R}^{3n}$  be any generic map. Then  $w_n^2(M^{2n}) + t(h) + l(h) = 0$  (in  $\mathbb{Z}_2$ ).*

*Proof.* Lemma 9 is proved in [24]. □

It follows from Lemma 9 and the fact that  $L(\partial g)$  changes at triple point instances of generic regular homotopies (see [6]), that the right-hand side of (15) is independent of  $g$ . Hence, the right-hand side of (15) induces a homomorphism  $\mathbf{Imm}(S^{8k+1}, \mathbb{R}^{8k+3}) \rightarrow \mathbb{Z}_2$  which is either zero (Equation (16) is true) or the Smale invariant (Equation (15) is true).

Thus, to prove (15) it is enough to find one example for which the right-hand side of (15) does not vanish.

**Remark 17.** In the same way that Corollary 1 follows from Theorem 1, it would follow from Equation 15 that  $\mathbf{Imm}(S^{8k+1}, \mathbb{R}^{8k+3}) \rightarrow \mathbf{Imm}(S^{8k+1}, \mathbb{R}^{12k+2})$  is injective. For  $k = 1$  this might be the case. Indeed,  $\mathbf{Imm}(S^9, \mathbb{R}^{11}) = \mathbb{Z}_2$  and  $\mathbf{Imm}(S^9, \mathbb{R}^{14}) = \mathbb{Z}_2$ , see [22].

**9.3. A formula expressing Smale invariant equals zero.** There is only one regular homotopy class of immersions  $S^{8k+5} \rightarrow \mathbb{R}^{8k+7}$ . Hence, Lemma 9 and the fact that  $L$  changes under triple point instances of generic regular homotopies imply the following (with notation as in Section 9.1):

**Proposition 1.** *Let  $f: S^{8k+5} \rightarrow \mathbb{R}^{8k+7}$  be an immersion. Then*

$$w_{4k+3}^2[\hat{M}^{8k+6}] + t(g) + l(g) + L(\partial g) = 0,$$

where we use the notation introduced in Section 9.1. (Note that this is an equation in  $\mathbb{Z}_2$ .) □

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